

Tanaka formula for strictly stable processes

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Abstract

For symmetric Lévy processes, if the local times exist, the Tanaka formula has already constructed via the techniques in the potential theory by Salminen and Yor (2007). In this paper, we study the Tanaka formula for arbitrary strictly stable processes with index $\alpha \in (1, 2)$ including spectrally positive and negative cases in a framework of Itô's stochastic calculus. Our approach to the existence of local times for such processes is different from Bertoin (1996).

1 Introduction

First, we begin with the definition of a local time for a stochastic process $X = (X_t)_{t \geq 0}$.

Definition 1.1. A family of random variables $L = \{L_t^a : a \in \mathbb{R}, t \geq 0\}$ is a local time of X if, for any bounded Borel measurable function $f : \mathbb{R} \rightarrow [0, \infty)$ and $t \geq 0$,

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(a) L_t^a da \quad \text{a.s.},$$

which is called an occupation time formula.

It is known that there exist local times of Brownian motions (cf. Berman [3]) and necessary and sufficient conditions of existence for Lévy processes can be found in Bertoin [4]. They considered the local time as the Radon–Nikodym derivative of the occupation time measure μ_t defined, for any Borel measurable function $f : \mathbb{R} \rightarrow [0, \infty)$ and $t \geq 0$, by

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(a) \mu_t(da).$$

The existence of almost surely jointly continuous local times was studied by Trotter [11] for Brownian motions, and by Boylan [5] for stable processes with

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index $\alpha \in (1, 2)$, and necessary and sufficient conditions for the almost sure joint continuity of local times were given by Barlow [2].

Let $B = (B_t)_{t \geq 0}$ be a Brownian motion on \mathbb{R} . It is well known that the Tanaka formula for a Brownian motion holds:

$$|B_t - a| - |B_0 - a| = \int_0^t \text{sgn}(B_s - a) dB_s + L_t^a,$$

where L^a denotes the local time of the Brownian motion at level a , and for more details, see, e.g. Ikeda and Watanabe [7]. By Lévy's characterization, $\int_0^\cdot \text{sgn}(B_s - a) dB_s$ is another Brownian motion. By setting $a = 0$, we know the process $|B|$ is the reflection of the Brownian motion which means that $(|B|, L^0)$ is the solution of the Skorohod problem for the Brownian motion. By the Doob–Meyer decomposition, the local time L^a can be understood as the unique bounded variation process which is the difference of the positive submartingale $|B - a|$ and the martingale $\int_0^\cdot \text{sgn}(B_s - a) dB_s + |B_0 - a|$. Thus, in this paper we say that the Tanaka formula holds if the local time can be expressed as the difference of such processes because it has a gap between the Skorohod solution and local times in the case of jump processes. The Tanaka formula has been already studied by Yamada [13] for symmetric stable processes with index $\alpha \in (1, 2)$, and by Salminen and Yor [9] for symmetric Lévy processes when local times exist. On the other hand, Watanabe [12] and Engelbert, Kurenok and Zalinescu [6] focused on the solution of the Skorohod problem.

In [9], they constructed the Tanaka formula for symmetric Lévy process X by using the continuous resolvent density u^p :

$$v(X_t - a) = v(a) + M_t^a + L_t^a$$

where $v(x) := \lim_{p \rightarrow 0} (u^p(0) - u^p(x))$ which is called a renormalized zero resolvent, and M^a is a martingale. But the existence of the limit is not clear in the asymmetric case and the representation for the martingale part is not given. In [14] Yano obtained a renormalized zero resolvent for asymmetric Lévy process under some conditions, and it associates the Tanaka formula based on local times with a harmonic function for the killed process upon hitting zero because the local time of such processes at level zero becomes zero.

In this paper, we shall show the Tanaka formula for arbitrary strictly stable processes including spectrally and negative cases with index $\alpha \in (1, 2)$ using Itô's stochastic calculus. Although our formula might give an insight on a reflection problem of jump type processes, the martingale part in the formula is not suitable to become the same law of the original process. Thus, there exists a gap between the Skorohod solution and the existence of local times in case of jump processes.

In Section 2, we shall prepare the Itô formula, the infinitesimal generator and the moments about stable processes. In Section 3, we construct the Tanaka formula for strictly stable processes.

2 Preliminaries

Let $C_{1+,b}^n(\mathbb{R})$ be the elements of $C^n(\mathbb{R})$ such that all derivatives of any orders greater than or equal to 1 are bounded, and $C_c^n(\mathbb{R})$ be the functions in $C^n(\mathbb{R})$ having compact support. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions on \mathbb{R} .

A Lévy process $Y = (Y)_{t \geq 0}$ is characterized by the Levy–Khintchine representation given as

$$\mathbb{E}[e^{iuY_t}] = \exp \left[t \left(ibu - \frac{1}{2} au^2 + \int_{\mathbb{R}_0} \{e^{iuy} - 1 - iuy \mathbf{1}_{|y| \leq 1}\} \nu(dy) \right) \right],$$

for a drift parameter $b \in \mathbb{R}$, a Gaussian coefficient $a \geq 0$ and a Lévy measure ν on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ satisfying the following integrability condition:

$$\int_{\mathbb{R}_0} (|h|^2 \wedge 1) \nu(dh) < \infty.$$

Note that a Lévy process is characterized by the triplet (b, a, ν) .

In this paper, we are concerned with one-dimensional strictly stable process $X = (X_t)_{t \geq 0}$ with index $\alpha \in (1, 2)$. This process X is characterized by the triplet $(b_\alpha, 0, \nu_\alpha)$ where the Lévy measure ν_α is given by

$$\nu_\alpha(dh) = \begin{cases} c_+ |h|^{-\alpha-1} dh & \text{on } (0, \infty), \\ c_- |h|^{-\alpha-1} dh & \text{on } (-\infty, 0), \end{cases}$$

where c_+ and c_- are non-negative constants such that $c_+ + c_- > 0$, and the drift parameter b_α is given by

$$b_\alpha = - \int_{|h| > 1} h \nu_\alpha(dh) = - \frac{c_+ - c_-}{\alpha - 1}.$$

The Lévy–Khintchine representation for X is given as

$$\mathbb{E}[e^{iuX_t}] = e^{t\eta(u)}, \quad u \in \mathbb{R},$$

where the function η is the Lévy symbol of X given by

$$\eta(u) = -d|u|^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right) \right),$$

where $d > 0$ and $\beta \in [-1, 1]$ are given by

$$d = \frac{c_+ + c_-}{2c(\alpha)}, \quad \beta = \frac{c_+ - c_-}{c_+ + c_-}$$

with

$$c(\alpha) = \frac{1}{\pi} \Gamma(\alpha + 1) \sin\left(\frac{\pi\alpha}{2}\right)$$

and the left-continuous signum function denoted by

$$\text{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0. \end{cases}$$

By the Lévy–Itô decomposition (see, Applebaum [1, Theorem 2.4.16]), X can be represented by

$$\begin{aligned} X_t &= X_0 - \int_{|h|>1} h\nu_\alpha(dh)t + \int_0^t \int_{|h|\leq 1} h\tilde{N}(ds, dh) + \int_0^t \int_{|h|>1} hN(ds, dh) \\ &= X_0 + \int_0^t \int_{\mathbb{R}_0} h\tilde{N}(ds, dh), \end{aligned} \quad (2.1)$$

where N is the jump measure of X which is a Poisson random measure with the intensity measure $ds\nu_\alpha(dh)$ and the compensated measure \tilde{N} is defined by

$$\tilde{N}(ds, dh) := N(ds, dh) - ds\nu_\alpha(dh).$$

Since $\mathbb{E}[|X_t|] < \infty$, it follows from (2.1) that X is a martingale. From the Itô formula ([1, Theorem 4.4.7]), we have the formula for (2.1).

Proposition 2.1. *For each $f \in C_{1+,b}^2(\mathbb{R})$, the following formula holds:*

$$\begin{aligned} f(X_t) - f(X_0) &= - \int_0^t \int_{|h|>1} f'(X_s)h\nu_\alpha(dh)ds \\ &\quad + \int_0^t \int_{|h|>1} \{f(X_{s-} + h) - f(X_{s-})\}N(ds, dh) \\ &\quad + \int_0^t \int_{|h|\leq 1} \{f(X_{s-} + h) - f(X_{s-})\}\tilde{N}(ds, dh) \\ &\quad + \int_0^t \int_{|h|\leq 1} \{f(X_s + h) - f(X_s) - f'(X_s)h\}\nu_\alpha(dh)ds. \end{aligned}$$

Remark 2.2. Since $f \in C_{1+,b}^2(\mathbb{R})$, it follows from the mean value theorem that

$$\begin{aligned} \int_{|h|\leq 1} |f(x+h) - f(x)|^2 \nu_\alpha(dh) &= \int_{|h|\leq 1} \left| \int_0^1 f'(x+\theta h)h d\theta \right|^2 \nu_\alpha(dh) \\ &\leq K^2 \int_{|h|\leq 1} |h|^2 \nu_\alpha(dh) = \frac{K^2(c_+ + c_-)}{2 - \alpha}. \end{aligned}$$

where K is a positive constant such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. Thus, the integral $\int_0^t \int_{|h|\leq 1} \{f(X_{s-} + h) - f(X_{s-})\}\tilde{N}(ds, dh)$ with respect to the compensated Poisson random measure is a square integrable martingale.

Then, we can rewrite Proposition 2.1 to the following:

Proposition 2.3. For each $f \in C_{1+,b}^2(\mathbb{R})$, the following formula holds:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t \int_{\mathbb{R}_0} \{f(X_{s-} + h) - f(X_{s-})\} \tilde{N}(ds, dh) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \{f(X_s + h) - f(X_s) - f'(X_s)h\} \nu_\alpha(dh) ds. \end{aligned}$$

Proof. It is sufficient to check

$$\int_0^t \int_{|h|>1} |f(X_s + h) - f(X_s)| \nu_\alpha(dh) ds < \infty.$$

Since $f \in C_{1+,b}^2(\mathbb{R})$, it follows from the mean value theorem that

$$\begin{aligned} \int_{|h|>1} |f(x+h) - f(x)| \nu_\alpha(dh) &= \int_{|h|>1} \left| \int_0^1 f'(x + \theta h) h d\theta \right| \nu_\alpha(dh) \\ &\leq K \int_{|h|>1} |h| \nu_\alpha(dh) = \frac{K(c_+ + c_-)}{\alpha - 1}. \end{aligned}$$

where K is a positive constant such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. \square

Remark 2.4. The integral $\int_0^t \int_{|h|>1} \{f(X_{s-} + h) - f(X_{s-})\} \tilde{N}(ds, dh)$ with respect to the compensated Poisson random measure is a martingale by the same argument as the proof of Proposition 2.3.

We consider the operator as follows:

$$\mathcal{L}f(x) := \int_{\mathbb{R}_0} \{f(x+h) - f(x) - f'(x)h\} \nu_\alpha(dh)$$

for all $f \in C_{1+,b}^2(\mathbb{R})$ and $x \in \mathbb{R}$. The operator \mathcal{L} coincides with the infinitesimal generator of X on $\mathcal{S}(\mathbb{R})$, see Sato [10, Theorem 31.5]. Using the operator \mathcal{L} , the Itô formula (Proposition 2.3) can be rewritten as follows:

$$f(X_t) - f(X_0) = \int_0^t \int_{\mathbb{R}_0} \{f(X_{s-} + h) - f(X_{s-})\} \tilde{N}(ds, dh) + \int_0^t \mathcal{L}f(X_s) ds.$$

Using the Fourier transform of $f \in L^1(\mathbb{R})$ defined by

$$\mathcal{F}[f](u) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux} f(x) dx \quad \text{for } u \in \mathbb{R},$$

and the inverse Fourier transform of $f \in L^1(\mathbb{R})$ defined by

$$\mathcal{F}^{-1}[f](x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} f(u) du \quad \text{for } x \in \mathbb{R},$$

the operator \mathcal{L} on $\mathcal{S}(\mathbb{R})$ is also represented as follows:

Proposition 2.5 ([1, Theorem 3.3.3]). *For each $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$,*

$$\mathcal{L}f(x) = \mathcal{F}^{-1}[\eta(u)\mathcal{F}[f](u)](x).$$

We will use the following moments of stable processes.

Lemma 2.6 ([10, Example 25.10]). *Let X be a stable process of index $\alpha \in (0, 2)$ with the triplet $(b, 0, \nu_\alpha)$. Then, for $t \geq 0$,*

$$\mathbb{E}|X_t|^\gamma = \begin{cases} < \infty & \text{if } 0 < \gamma < \alpha, \\ = \infty & \text{if } \gamma \geq \alpha. \end{cases}$$

Moreover, we will use the following negative-order moments.

Lemma 2.7. *Let X be a stable processes of index $\alpha \in (0, 2)$ with the triplet $(b, 0, \nu_\alpha)$. Then, for all $t > 0$ and $x \in \mathbb{R}$,*

$$\mathbb{E}[|X_t - x|^{-\gamma}] \leq S(\alpha, \gamma)t^{-\gamma/\alpha} \quad \text{if } 0 < \gamma < 1,$$

where the constant $S(\alpha, \gamma)$ is independent of x .

Proof. By the monotone convergence theorem, we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[|X_t - x|^{-\gamma} e^{-\varepsilon|X_t - x|}] = \mathbb{E}[|X_t - x|^{-\gamma}].$$

Since $|e^{t\eta(\cdot)}| = e^{-dt|\cdot|^\alpha} \in L^1(\mathbb{R})$ for $t > 0$, transition density p_t is given by

$$p_t(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuy} e^{t\eta(u)} du,$$

for each $t > 0$ and $y \in \mathbb{R}$. Thus, it follows from Fubini's theorem that for all $\varepsilon > 0$,

$$\begin{aligned} \mathbb{E}[|X_t - x|^{-\gamma} e^{-\varepsilon|X_t - x|}] &= \int_{\mathbb{R}} |y - x|^{-\gamma} e^{-\varepsilon|y - x|} p_t(y) dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{t\eta(u)} \int_{\mathbb{R}} |y - x|^{-\gamma} e^{-\varepsilon|y - x|} e^{-iuy} dy du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{t\eta(u) - iux} \int_{\mathbb{R}} |z|^{-\gamma} e^{-\varepsilon|z| - iuz} dz du, \end{aligned}$$

since $|\cdot|^{-\gamma} e^{-\varepsilon|\cdot|} \in L^1(\mathbb{R})$.

Now, we will make use of the following identity:

$$\int_0^\infty x^{\xi-1} e^{-xz} dx = \Gamma(\xi) z^{-\xi} \quad (2.2)$$

for all $\xi > 0$ and $\Re z > 0$, where $\Re z$ is the real part of z . By $1 - \gamma > 0$ and $\Re(\varepsilon \pm iu) = \varepsilon > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}} |z|^{-\gamma} e^{-\varepsilon|z| - iuz} dz &= \int_0^\infty z^{-\gamma} e^{-(\varepsilon + iu)z} dz + \int_0^\infty z^{-\gamma} e^{-(\varepsilon - iu)z} dz \\ &= \Gamma(1 - \gamma)(\varepsilon + iu)^{\gamma-1} + \Gamma(1 - \gamma)(\varepsilon - iu)^{\gamma-1}. \end{aligned}$$

We then have for $u \neq 0$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \{(\varepsilon + iu)^{\gamma-1} + (\varepsilon - iu)^{\gamma-1}\} &= (iu)^{\gamma-1} + (-iu)^{\gamma-1} \\ &= 2|u|^{\gamma-1} \cos\left(\frac{\pi(\gamma-1)}{2}\right). \end{aligned}$$

By $-1 < \gamma - 1 < 0$, we have

$$|\varepsilon \pm iu|^{\gamma-1} |e^{t\eta(u)-iux}| \leq |u|^{\gamma-1} e^{-dt|u|^\alpha} \in L^1(\mathbb{R}).$$

Hence, it follows from the dominated convergence theorem that for all $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[|X_t - x|^{-\gamma} e^{-\varepsilon|X_t - x|} \right] &= \frac{\Gamma(1-\gamma)}{2\pi} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \{(\varepsilon + iu)^{\gamma-1} + (\varepsilon - iu)^{\gamma-1}\} e^{t\eta(u)-iux} du \\ &= \frac{\Gamma(1-\gamma)}{\pi} \cos\left(\frac{\pi(\gamma-1)}{2}\right) \int_{\mathbb{R}} |u|^{\gamma-1} e^{t\eta(u)-iux} du \\ &\leq \frac{\Gamma(1-\gamma)}{\pi} \cos\left(\frac{\pi(\gamma-1)}{2}\right) \int_{\mathbb{R}} |u|^{\gamma-1} e^{-dt|u|^\alpha} du \\ &= \frac{\Gamma(1-\gamma)t^{-\gamma/\alpha}}{\pi} \cos\left(\frac{\pi(\gamma-1)}{2}\right) \int_{\mathbb{R}} |v|^{\gamma-1} e^{-d|v|^\alpha} dv < \infty. \end{aligned}$$

The proof is now complete. \square

In order to use the continuity about the law of X_t , we check the following condition.

Lemma 2.8 ([10, Theorem 27.4]). *Let Y be a Lévy process with the triplet (b, a, ν) . The followings are equivalent.*

- (i) $\mathbb{P}(Y_t = y) = 0$ for all $y \in \mathbb{R}$ and $t > 0$.
- (ii) $a \neq 0$ or $\nu(\mathbb{R}_0) = \infty$.

Thus, by $\nu_\alpha(\mathbb{R}_0) = \infty$, we have that the law of X_t is continuous for all $t > 0$.

3 Main results

Before we establish the Tanaka formula for general strictly stable processes, we need the following lemma. In case of $\beta = 0$, that is, X is a symmetric stable process with index $\alpha \in (1, 2)$, it is shown in Komatsu [8] the following result:

Lemma 3.1. *Let*

$$F(x) := D(\alpha)(1 - \beta \operatorname{sgn}(x))|x|^{\alpha-1}$$

where

$$D(\alpha) = \frac{c(-\alpha)}{d(1 + \beta^2 \tan^2(\pi\alpha/2))}.$$

Then, we have for all $\phi \in C_c^\infty(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\mathcal{L}(F * \phi)(x) = \phi(x)$$

where $F * \phi$ is given by the convolution of F and ϕ :

$$(F * \phi)(x) := \int_{\mathbb{R}} F(y)\phi(x-y)dy.$$

Proof. Let $F_\phi := F * \phi$ for $\phi \in C_c^\infty(\mathbb{R})$. We know $F_\phi \in C^\infty(\mathbb{R})$. Since we have for all $n \in \mathbb{N}$,

$$F_\phi^{(n)}(x) = \int_{\mathbb{R}} F(y)\phi^{(n)}(x-y)dy = \int_{\mathbb{R}} F'(y)\phi^{(n-1)}(x-y)dy,$$

where

$$F'(y) = (\alpha - 1)D(\alpha)(\operatorname{sgn}(y) - \beta)|y|^{\alpha-2},$$

we have $F_\phi^{(n)}$ is bounded for all $n \in \mathbb{N}$. Thus, we have $F_\phi \in C_{1+,b}^\infty(\mathbb{R})$.

Set $F_\varepsilon(x) := F(x)e^{-\varepsilon|x|}$ for $\varepsilon > 0$ and $F_{\varepsilon,\phi} := F_\varepsilon * \phi \in C^\infty(\mathbb{R})$. Since $1 + |x|^2 \leq 2(1 + |x-y|^2)(1 + |y|^2)$ for all $x, y \in \mathbb{R}$, we have for all $k, n \in \mathbb{N}$,

$$\begin{aligned} & \sup_{x \in \mathbb{R}} (1 + |x|^2)^k |F_{\varepsilon,\phi}^{(n)}(x)| \\ & \leq 2^k \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} (1 + |x-y|^2)^k (1 + |y|^2)^k |F_\varepsilon(y)\phi^{(n)}(x-y)|dy \\ & \leq 2^{k+1} D(\alpha) \sup_{z \in \mathbb{R}} (1 + |z|^2)^k |\phi^{(n)}(z)| \int_{\mathbb{R}} (1 + |y|^2)^k |y|^{\alpha-1} e^{-\varepsilon|y|} dy < \infty, \end{aligned}$$

since $\phi \in C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. Thus, we have $F_{\varepsilon,\phi} \in \mathcal{S}(\mathbb{R})$. By Proposition 2.5, we have

$$\begin{aligned} & \int_{\mathbb{R}_0} \{F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h\}dh \\ & = \mathcal{F}^{-1}[\eta(u)\mathcal{F}[F_{\varepsilon,\phi}](u)](x) \\ & = \sqrt{2\pi}\mathcal{F}^{-1}[\eta(u)\mathcal{F}[F_\varepsilon](u)\mathcal{F}[\phi](u)](x). \end{aligned}$$

Since $|x|e^{-|x|} \leq 1$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} |F'_\varepsilon(x)| &= \left| (\alpha - 1)D(\alpha)(\operatorname{sgn}(x) - \beta)|x|^{\alpha-2}e^{-\varepsilon|x|} \right. \\ & \quad \left. - \varepsilon D(\alpha)(\operatorname{sgn}(x) - \beta)|x|^{\alpha-1}e^{-\varepsilon|x|} \right| \\ & \leq 2D(\alpha)|x|^{\alpha-2}e^{-\varepsilon|x|} + 2\varepsilon D(\alpha)|x|^{\alpha-1}e^{-\varepsilon|x|} \\ & \leq 4D(\alpha)|x|^{\alpha-2}. \end{aligned}$$

By using integration by parts, it follows from Fubini's theorem that

$$\begin{aligned}
& |F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h| \\
&= \left| \int_{\mathbb{R}} F_{\varepsilon}(y) \{ \phi(x+h-y) - \phi(x-y) - \phi'(x-y)h \} dy \right| \\
&= \left| \int_{\mathbb{R}} F'_{\varepsilon}(y)h \int_0^1 \{ \phi(x+\theta h-y) - \phi(x-y) \} d\theta dy \right| \\
&= \left| \int_0^1 \int_{\mathbb{R}} F'_{\varepsilon}(y)h \{ \phi(x+\theta h-y) - \phi(x-y) \} dy d\theta \right| \\
&\leq 2C_1(\alpha)|h|,
\end{aligned}$$

where the constant $C_1(\alpha)$ is given by

$$C_1(\alpha) = 4D(\alpha) \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |y|^{\alpha-2} |\phi(x-y)| dy.$$

Similarly, it follows that

$$\begin{aligned}
& |F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h| \\
&= \left| \int_{\mathbb{R}} F'_{\varepsilon}(y)h^2 \int_0^1 \phi'(x+\theta h-y)(1-\theta) d\theta dy \right| \\
&= \left| \int_0^1 \int_{\mathbb{R}} F'_{\varepsilon}(y)h^2 \phi'(x+\theta h-y)(1-\theta) dy d\theta \right| \\
&\leq \frac{C_2(\alpha)|h|^2}{2},
\end{aligned}$$

where the constant $C_2(\alpha)$ is given by

$$C_2(\alpha) = 4D(\alpha) \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |y|^{\alpha-2} |\phi'(x-y)| dy.$$

By $1 < \alpha < 2$, we know

$$\int_{\mathbb{R}_0} (|h|^2 \wedge |h|) \nu_{\alpha}(dh) = \frac{c_+ + c_-}{2-\alpha} + \frac{c_+ + c_-}{\alpha-1} < \infty.$$

Hence, it follows from the dominated convergence theorem that

$$\begin{aligned}
& \lim_{\varepsilon \downarrow 0} \sqrt{2\pi} \mathcal{F}^{-1}[\eta(u) \mathcal{F}[F_{\varepsilon}](u) \mathcal{F}[\phi](u)](x) \\
&= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}_0} \{ F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h \} dh \\
&= \int_{\mathbb{R}_0} \{ F_{\phi}(x+h) - F_{\phi}(x) - F'_{\phi}(x)h \} dh.
\end{aligned}$$

Now, by using (2.2), we have

$$\mathcal{F}[|x|^{\alpha-1} e^{-\varepsilon|x|}](u) = \frac{\Gamma(\alpha)}{\sqrt{2\pi}} \left((\varepsilon + iu)^{-\alpha} + (\varepsilon - iu)^{-\alpha} \right),$$

and

$$\mathcal{F}[|x|^{\alpha-1} \operatorname{sgn}(x) e^{-\varepsilon|x|}](u) = \frac{\Gamma(\alpha)}{\sqrt{2\pi}} ((\varepsilon + iu)^{-\alpha} - (\varepsilon - iu)^{-\alpha}).$$

We then have for $u \neq 0$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathcal{F}[|x|^{\alpha-1} e^{-\varepsilon|x|}](u) &= \frac{\Gamma(\alpha)}{\sqrt{2\pi}} |u|^{-\alpha} (i^{-\alpha} + (-i)^{-\alpha}) \\ &= \frac{2\Gamma(\alpha)}{\sqrt{2\pi}} |u|^{-\alpha} \cos\left(\frac{\pi\alpha}{2}\right), \end{aligned}$$

and

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathcal{F}[|x|^{\alpha-1} e^{-\varepsilon|x|} \operatorname{sgn}(x)](u) &= \frac{\Gamma(\alpha)}{\sqrt{2\pi}} |u|^{-\alpha} \operatorname{sgn}(u) (i^{-\alpha} - (-i)^{-\alpha}) \\ &= -i \frac{2\Gamma(\alpha)}{\sqrt{2\pi}} |u|^{-\alpha} \operatorname{sgn}(u) \sin\left(\frac{\pi\alpha}{2}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \eta(u) \mathcal{F}[F_\varepsilon](u) \\ &= \left(-c(-\alpha) \frac{d|u|^\alpha (1 - i\beta \operatorname{sgn}(u) \tan(\pi\alpha/2))}{d(1 + \beta^2 \tan^2(\pi\alpha/2))} \right) \\ &\quad \times \lim_{\varepsilon \downarrow 0} \left(\mathcal{F}[|x|^{\alpha-1} e^{-\varepsilon|x|}](u) - \beta \mathcal{F}[|x|^{\alpha-1} e^{-\varepsilon|x|} \operatorname{sgn}(x)](u) \right) \\ &= \left(-\frac{c(-\alpha)|u|^\alpha}{1 + i\beta \operatorname{sgn}(u) \tan(\pi\alpha/2)} \right) \\ &\quad \times \frac{2\Gamma(\alpha)}{\sqrt{2\pi}} |u|^{-\alpha} \left(\cos\left(\frac{\pi\alpha}{2}\right) + i\beta \operatorname{sgn}(u) \sin\left(\frac{\pi\alpha}{2}\right) \right) \\ &= -\frac{2\Gamma(\alpha)c(-\alpha)}{\sqrt{2\pi}} \cos\left(\frac{\pi\alpha}{2}\right) = \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

By $\phi \in C_c^\infty(\mathbb{R})$, we have

$$\left| \frac{u}{\varepsilon \pm iu} \right|^\alpha \mathcal{F}[\phi](u) \leq \mathcal{F}[\phi](u) \in C_c^\infty(\mathbb{R}).$$

Hence, it follows from the dominated convergence theorem that

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \sqrt{2\pi} \mathcal{F}^{-1}[\eta(u) \mathcal{F}[F_\varepsilon](u) \mathcal{F}[\phi](u)](x) \\ &= \sqrt{2\pi} \mathcal{F}^{-1} \left[\lim_{\varepsilon \downarrow 0} \eta(u) \mathcal{F}[F_\varepsilon](u) \mathcal{F}[\phi](u) \right](x) \\ &= \mathcal{F}^{-1}[\mathcal{F}[\phi](u)](x) = \phi(x). \end{aligned}$$

The proof is now complete. \square

Remark 3.2. A strictly stable process with index $\alpha \in (0, 1)$ is characterized by $(b'_\alpha, 0, \nu_\alpha)$ where $b'_\alpha = \int_{|h| \leq 1} h \nu_\alpha(dh) = (c_+ - c_-)/(1 - \alpha)$. In this case, the operator corresponding to \mathcal{L} can be represented as

$$\mathcal{L}' f(x) = \int_{\mathbb{R}_0} \{f(x+h) - f(x)\} \nu_\alpha(dh)$$

for $f \in C_{1+,b}^1(\mathbb{R})$ which is bounded. Then, the Lemma 3.1 holds for \mathcal{L}' . But the local time does not exist if $\alpha \in (0, 1)$. Indeed, in [4, Theorem V.1], the condition

$$\int_{\mathbb{R}} \Re \left(\frac{1}{1 - \eta(u)} \right) du < \infty \quad (3.1)$$

holds if and only if the occupation time measure μ_t is absolutely continuous with respect to the Lebesgue measure such that its density is in $L^2(da \otimes d\mathbb{P})$. Moreover, if (3.1) fails, then the measure μ_t is singular for every $t > 0$, almost surely. In case of $\alpha \in (0, 1)$, since (3.1) does not hold, the measure μ_t is singular.

We introduce a mollifier as follows:

Definition 3.3. The function ρ is a mollifier if, the function ρ on \mathbb{R} satisfies

- (i) $\rho(x) \geq 0$ for all $x \in \mathbb{R}$.
- (ii) $\rho \in C_c^\infty(\mathbb{R})$.
- (iii) $\text{supp} \rho = [-1, 1]$.
- (iv) $\int_{\mathbb{R}} \rho(x) dx = 1$.

Furthermore, we define $\rho_n(x) := n\rho(nx)$ for all $n \in \mathbb{N}$. A family of functions $(\rho_n)_{n \in \mathbb{N}}$ satisfies that $\rho_n \rightarrow \delta_0$ as $n \rightarrow \infty$ where δ_0 is the Dirac delta function, in the sense of Schwartz distributions, that means

$$\left| \int_{\mathbb{R}} \rho_n(x) \phi(x) dx - \phi(0) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

Now let us state our main theorem which we call the Tanaka formula for arbitrary strictly stable processes with index $\alpha \in (1, 2)$.

Theorem 3.4. *Let F be the same as in Lemma 3.1. Then, there exists the local time $L = \{L_t^a : a \in \mathbb{R}, t \geq 0\}$ of X which is given by*

$$L_t^a = F(X_t - a) - F(X_0 - a) - M_t^a \quad \text{for } a \in \mathbb{R}, t \geq 0,$$

where the process $(M_t^a)_{t \geq 0}$ given by

$$M_t^a = \int_0^t \int_{\mathbb{R}_0} \{F(X_{s-} - a + h) - F(X_{s-} - a)\} \tilde{N}(ds, dh)$$

is a square integrable martingale.

Remark 3.5. The local time can be also represented by

$$L_t^a = \lim_{n \rightarrow \infty} \int_0^t \rho_n(X_s - a) ds \quad \text{in } L^2(d\mathbb{P}),$$

where $(\rho_n)_{n \in \mathbb{N}}$ is given by $\rho_n(x) = n\rho(nx)$ for all $n \in \mathbb{N}$ with a mollifier ρ .

Proof of Theorem 3.4. Let $F_n = F * \rho_n$ for all $n \in \mathbb{N}$. By the argument in the proof of Lemma 3.1, we have $F_n \in C_{1+,b}^\infty(\mathbb{R})$. By the Itô formula (Proposition 2.3), we have for all $t \geq 0$ and $a \in \mathbb{R}$,

$$F_n(X_t - a) - F_n(X_0 - a) = M_t^{a,n} + V_t^{a,n},$$

where

$$M_t^{a,n} = \int_0^t \int_{\mathbb{R}_0} \{F_n(X_{s-} - a + h) - F_n(X_{s-} - a)\} \tilde{N}(ds, dh)$$

and

$$V_t^{a,n} = \int_0^t \mathcal{L}F_n(X_s - a) ds.$$

First we will show that $F_n(X_t - a), F(X_t - a) \in L^2(d\mathbb{P})$ and $F_n(X_t - a) \rightarrow F(X_t - a)$ in $L^2(d\mathbb{P})$ as $n \rightarrow \infty$. Using the following inequality:

$$|x + y|^{\alpha-1} \leq |x|^{\alpha-1} + |y|^{\alpha-1} \quad \text{for all } x, y \in \mathbb{R},$$

we have for $x \in \mathbb{R}$,

$$\begin{aligned} 0 \leq F_n(x) &= \int_{\mathbb{R}} F(x - y) \rho_n(y) dy \\ &\leq \int_{-1/n}^{1/n} 2D(\alpha) (|x|^{\alpha-1} + |y|^{\alpha-1}) \rho_n(y) dy \leq 2D(\alpha) (|x|^{\alpha-1} + 1). \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} \mathbb{E}[|F_n(X_t - a)|^2] &\leq 4D(\alpha)^2 \mathbb{E}[(|X_t - a|^{\alpha-1} + 1)^2] \\ &\leq 12D(\alpha)^2 (\mathbb{E}[|X_t|^{2\alpha-2}] + |a|^{2\alpha-2} + 1) < \infty, \end{aligned}$$

since $0 < 2\alpha - 2 < \alpha$. Similarly, we have

$$\begin{aligned} \mathbb{E}[|F(X_t - a)|^2] &\leq 4D(\alpha)^2 \mathbb{E}[(|X_t|^{\alpha-1} + |a|^{\alpha-1})^2] \\ &\leq 8D(\alpha)^2 (\mathbb{E}[|X_t|^{2\alpha-2}] + |a|^{2\alpha-2}) < \infty. \end{aligned}$$

Hence, it follows from the dominated convergence theorem that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E}[|F_n(X_t - a) - F(X_t - a)|^2] \\ &= \mathbb{E} \left[\lim_{n \rightarrow \infty} |F_n(X_t - a) - F(X_t - a)|^2 \right] = 0. \end{aligned} \tag{3.2}$$

Next, we will show that $M_t^{a,n}$ and M_t^a are square integrable martingales, and $M_t^{a,n} \rightarrow M_t^a$ in $L^2(d\mathbb{P})$ as $n \rightarrow \infty$. Since $||x+1|^{\alpha-1} - 1| \leq |x|$ for all $x \in \mathbb{R}$, we have for all $0 \leq \varepsilon \leq 2 - \alpha$,

$$\begin{aligned} ||x+1|^{\alpha-1} - 1| &\leq |x|^{(\alpha+\varepsilon)/2} \wedge |x|^{\alpha-1}, \\ ||x+1|^{\alpha-1} \operatorname{sgn}(x+1) - 1| &\leq 3 \left(|x|^{(\alpha+\varepsilon)/2} \wedge |x|^{\alpha-1} \right), \end{aligned}$$

by $\alpha - 1 < (\alpha + \varepsilon)/2 \leq 1$. Thus, we have for $x \neq 0$ and $y \in \mathbb{R}$,

$$\begin{aligned} |F(x+y) - F(x)|^2 &= |D(\alpha)\{|x+y|^{\alpha-1} - \beta|x+y|^{\alpha-1} \operatorname{sgn}(x+y)\} \\ &\quad - D(\alpha)\{|x|^{\alpha-1} - \beta|x|^{\alpha-1} \operatorname{sgn}(x)\}|^2 \\ &\leq 2D(\alpha)^2 ||x+y|^{\alpha-1} - |x|^{\alpha-1}|^2 \\ &\quad + 2D(\alpha)^2 ||x+y|^{\alpha-1} \operatorname{sgn}(x+y) - |x|^{\alpha-1} \operatorname{sgn}(x)|^2 \\ &= 2D(\alpha)^2 |x|^{2\alpha-2} \left| 1 + \frac{y}{x} \right|^{\alpha-1} - 1 \Big|^2 \\ &\quad + 2D(\alpha)^2 |x|^{2\alpha-2} \left| \left| 1 + \frac{y}{x} \right|^{\alpha-1} \operatorname{sgn} \left(1 + \frac{y}{x} \right) - 1 \right|^2 \\ &\leq 8D(\alpha)^2 (|x|^{\alpha-\varepsilon-2} |y|^{\alpha+\varepsilon} \wedge |y|^{2\alpha-2}) \end{aligned}$$

Now, choose ε_0 such that $0 < \varepsilon_0 < (\alpha - 1) \wedge (2 - \alpha)$. Since the law of X_t is continuous for all $t > 0$ by Lemma 2.8, it follows from the Cauchy–Schwarz inequality, Fubini’s theorem and Lemma 2.7 that for all $s > 0$,

$$\begin{aligned} &\mathbb{E} [|F_n(X_s + h - a) - F_n(X_s - a)|^2] \\ &\leq \mathbb{E} \left[\int_{\mathbb{R}} \rho_n(y) |F(X_s + h - a - y) - F(X_s - a - y)|^2 dy \right] \\ &\leq 8D(\alpha)^2 \int_{\mathbb{R}} \rho_n(y) |h|^{2\alpha-2} \mathbf{1}_{\{|h|>1\}} dy \\ &\quad + 8D(\alpha)^2 \int_{\mathbb{R}} \rho_n(y) \mathbb{E} [|X_s - a - y|^{\alpha-\varepsilon_0-2} |h|^{\alpha+\varepsilon_0}] \mathbf{1}_{\{|h|\leq 1\}} dy \\ &\leq 8D(\alpha)^2 |h|^{2\alpha-2} \mathbf{1}_{\{|h|>1\}} \\ &\quad + 8D(\alpha)^2 S(\alpha, -\alpha + \varepsilon_0 + 2) s^{(\alpha-\varepsilon_0-2)/\alpha} |h|^{\alpha+\varepsilon_0} \mathbf{1}_{\{|h|\leq 1\}}, \end{aligned}$$

by $0 < -\alpha + \varepsilon_0 + 2 < 1$. Similarly, it follows that for all $s > 0$,

$$\begin{aligned} &\mathbb{E} [|F(X_s + h - a) - F(X_s - a)|^2] \\ &\leq 8D(\alpha)^2 |h|^{2\alpha-2} \mathbf{1}_{\{|h|>1\}} \\ &\quad + 8D(\alpha)^2 \mathbb{E} [|X_s - a|^{\alpha-\varepsilon_0-2} |h|^{\alpha+\varepsilon_0}] \mathbf{1}_{\{|h|\leq 1\}} \\ &\leq 8D(\alpha)^2 |h|^{2\alpha-2} \mathbf{1}_{\{|h|>1\}} \\ &\quad + 8D(\alpha)^2 S(\alpha, 2 + \varepsilon_0 - \alpha) s^{(\alpha-\varepsilon_0-2)/\alpha} |h|^{\alpha+\varepsilon_0} \mathbf{1}_{\{|h|\leq 1\}}. \end{aligned}$$

By $0 < \varepsilon_0 < \alpha - 1$ and $1 < \alpha < 2$, we know

$$\int_{\mathbb{R}_0} (|h|^{\alpha+\varepsilon_0} \wedge |h|^{2\alpha-2}) \nu_\alpha(dh) = \frac{c_+ + c_-}{\varepsilon_0} + \frac{c_+ + c_-}{2 - \alpha} < \infty,$$

and

$$\int_0^t s^{(\alpha-\varepsilon_0-2)/\alpha} ds = \frac{\alpha}{2\alpha - \varepsilon_0 - 2} t^{(2\alpha-\varepsilon_0-2)/\alpha} < \infty.$$

Hence, it follows that $M_t^{a,n}$ and M_t^a are square integrable martingales, and from the dominated convergence theorem and (3.2) that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}[|M_t^{a,n} - M_t^a|^2] \\ &= \lim_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}_0} \mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) \\ &\quad - \{F(X_s + h - a) - F(X_s - a)\}|^2] \nu_\alpha(dh) ds \\ &= \int_0^t \int_{\mathbb{R}_0} \lim_{n \rightarrow \infty} \mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) \\ &\quad - \{F(X_s + h - a) - F(X_s - a)\}|^2] \nu_\alpha(dh) ds \\ &= 0. \end{aligned} \tag{3.3}$$

Finally, we will show that $L_t^a := F(X_t - a) - F(X_0 - a) - M_t^a$ is the local time of X . It is sufficient to show that an occupation time formula holds for $g \in C_c(\mathbb{R})$. By Lemma 3.1 and Fubini's theorem, we have for a.e.- ω ,

$$\begin{aligned} \int_{\mathbb{R}} g(a) V_t^{a,n}(\omega) da &= \int_{\mathbb{R}} g(a) \int_0^t \rho_n(X_s(\omega) - a) ds da \\ &= \int_0^t (g * \rho_n)(X_s(\omega)) ds. \end{aligned}$$

Thus, we will show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g(a) V_t^{a,n} da = \int_{\mathbb{R}} g(a) L_t^a da \quad \text{in } L^2(d\mathbb{P}),$$

and

$$\lim_{n \rightarrow \infty} \int_0^t (g * \rho_n)(X_s) ds = \int_0^t g(X_s) ds \quad \text{in } L^2(d\mathbb{P}).$$

By using the Cauchy-Schwarz inequality and Fubini's theorem, it follows from

the dominated convergence theorem, (3.2) and (3.3) that for $g \in C_c(\mathbb{R})$,

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_{\mathbb{R}} g(a) V_t^{a,n} da - \int_{\mathbb{R}} f(a) L_t^a da \right|^2 \right] \\
& \leq \mathbb{E} \left[\int_{\mathbb{R}} |g(a)| da \int_{\mathbb{R}} |g(a)| |V_t^{a,n} - L_t^a|^2 da \right] \\
& \leq \int_{\mathbb{R}} |g(a)| da \int_{\mathbb{R}} |g(a)| \mathbb{E} [|F_n(X_t - a) - F(X_t - a)|^2] da \\
& \quad + \int_{\mathbb{R}} |g(a)| da \int_{\mathbb{R}} |g(a)| \mathbb{E} [|F_n(X_0 - a) - F(X_0 - a)|^2] da \\
& \quad + \int_{\mathbb{R}} |g(a)| da \int_{\mathbb{R}} |g(a)| \mathbb{E} [|M_t^{a,n} - M_t^a|^2] da \\
& \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

By using the Cauchy-Schwarz inequality and Fubini's theorem, it follows from the dominated convergence theorem that for $g \in C_c(\mathbb{R})$,

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^t (g * \rho_n)(X_s) ds - \int_0^t g(X_s) ds \right|^2 \right] \\
& \leq t \mathbb{E} \left[\int_0^t \left| \int_{\mathbb{R}} \rho_n(y) \{g(X_s - y) - g(X_s)\} dy \right|^2 ds \right] \\
& \leq t \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} \rho_n(y) |g(X_s - y) - g(X_s)|^2 dy ds \right] \\
& = t \int_0^t \int_{\mathbb{R}} n \rho(ny) \mathbb{E} [|g(X_s - y) - g(X_s)|^2] dy ds \\
& = t \int_0^t \int_{\mathbb{R}} \rho(z) \mathbb{E} [|g(X_s - n^{-1}z) - g(X_s)|^2] dz ds \\
& \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

The proof is now complete. \square

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